THE BUCHSBAUM-RIM FUNCTION OF A PARAMETER MODULE

FUTOSHI HAYASAKA

Abstract. This note is basically a summary of a part of the paper [11] with Eero Hyry (University of Tampere). In this note we prove that the Buchsbaum-Rim function \( \lambda_A(S_{\nu+1}(F)/N^{\nu+1}) \) of a parameter module \( N \) in \( F \) is bounded above by \( e(F/N)(\nu+d+r-1) \) for every integer \( \nu \geq 0 \). Moreover, it turns out that the base ring \( A \) is Cohen-Macaulay once the equality holds for some integer \( \nu \). As a direct consequence, we observe that the first Buchsbaum-Rim coefficient \( e_1(F/N) \) of a parameter module \( N \) is always non-positive.

1. Introduction

Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d \). Let \( F = A^r \) be a free module of rank \( r > 0 \), and let \( S = S_A(F) \) be the symmetric algebra of \( F \), which is a polynomial ring over \( A \). For a submodule \( M \) of \( F \), let \( \mathcal{R}(M) \) denote the image of the natural homomorphism \( S_A(M) \to S_A(F) \), which is a standard graded subalgebra of \( S \). Assume that the quotient \( F/M \) has finite length and \( M \subseteq \mathfrak{m}F \). Then we can consider the function

\[
\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} ; \quad \nu \mapsto \ell_A(S_{\nu}/M^{\nu})
\]

where \( S_{\nu} \) and \( M^{\nu} \) denote the homogeneous components of degree \( \nu \) of \( S \) and \( \mathcal{R}(M) \), respectively. Buchsbaum and Rim studied this function in [4] in order to generalize the notion of the usual Hilbert-Samuel multiplicity of an \( \mathfrak{m} \)-primary ideal. They proved that \( \lambda(\nu) \) eventually coincides with a polynomial \( P(\nu) \) of degree \( d + r - 1 \). This polynomial can then be written in the form

\[
P(\nu) = \sum_{i=0}^{d+r-1} (-1)^i e_i(F/M) \binom{\nu + d + r - 2 - i}{d + r - 1 - i}
\]

with integer coefficients \( e_i(F/M) \). The coefficients \( e_i(F/M) \) are called the Buchsbaum-Rim coefficients of \( F/M \). The Buchsbaum-Rim multiplicity of \( F/M \), denoted by \( e(F/M) \), is now defined to be the leading coefficient \( e_0(F/M) \).

In their article Buchsbaum and Rim also introduced the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module \( N \) in \( F \) is said to be a parameter module in \( F \), if the following three conditions are satisfied: (i) \( F/N \) has finite length, (ii) \( N \subseteq \mathfrak{m}F \), and (iii) \( \mu_A(N) = d + r - 1 \), where \( \mu_A(N) \) is the minimal number of generators of \( N \).

A starting point of this note is the characterization of the Cohen-Macaulay property of \( A \) given in [4, Corollary 4.5] by means of the equality \( \ell_A(F/N) = e(F/N) \) for every

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parameter module $N$ of rank $r$ in $F = A^r$. Brennan, Ulrich and Vasconcelos observed in [1, Theorem 3.4] that if $A$ is Cohen-Macaulay, then in fact

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\left(\frac{\nu + d + r - 1}{d + r - 1}\right)$$

for all integers $\nu \geq 0$. Our main result is now as follows:

**Theorem 1.** Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$.

1. For any rank $r > 0$, the inequality

$$\ell_A(S_{\nu+1}/N^{\nu+1}) \geq e(F/N)\left(\frac{\nu + d + r - 1}{d + r - 1}\right)$$

always holds true for every parameter module $N$ in $F = A^r$ and for every integer $\nu \geq 0$.

2. The following statements are equivalent:
   (i) $A$ is a Cohen-Macaulay local ring;
   (ii) There exists an integer $r > 0$ and a parameter module $N$ of rank $r$ in $F = A^r$ such that the equality

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\left(\frac{\nu + d + r - 1}{d + r - 1}\right)$$

holds true for some integer $\nu \geq 0$.

This generalizes our previous result [10, Theorem 1.3] where we assumed that $\nu = 0$. The equivalence of (i) and (ii) in (2) seems to contain some new information even in the ideal case. Indeed, it improves a recent observation that the ring $A$ is Cohen-Macaulay if there exists a parameter ideal $Q$ in $A$ such that $\ell_A(A/Q^{\nu+1}) = e(A/Q)(\nu + d)$ for all $\nu \gg 0$ (see [8, 12]). Moreover, as a direct consequence of (1), we have the non-positivity of the first Buchsbaum-Rim coefficient of a parameter module.

**Corollary 2.** For any rank $r > 0$, the inequality

$$e_1(F/N) \leq 0$$

always holds true for every parameter module $N$ in $F = A^r$.

Mandal and Verma have recently proved that $e_1(A/Q) \leq 0$ for any parameter ideal $Q$ in $A$ (see [15], and also [8]). Corollary 2 can be viewed as the module version of this fact. However, our proof based on the inequality in Theorem 1 (1) is completely different from theirs and is considerably more simpler.

2. Preliminaries

Let $(A, m)$ be a Noetherian local ring of dimension $d$. Let $F = A^r$ be a free module of rank $r > 0$. Let $S = S_A(F)$ be the symmetric algebra of $F$. Let $N$ be a parameter module in $F$, that is, $N$ is a submodule of $F$ satisfying the conditions: (i) $\ell_A(F/N) < \infty$, (ii) $N \subseteq mF$, and (iii) $\mu_A(N) = d + r - 1$. We put $n = d + r - 1$. Let $N^\nu$ be the homogeneous
component of degree $\nu$ of the standard graded subalgebra $\mathcal{R}(N) = \text{Im}(S_A(N) \to S)$ of $S$. Let $\tilde{N} = (c_{ij})$ be the matrix associated to a minimal free presentation

$$A^n \xrightarrow{\tilde{N}} F \to F/N \to 0$$

of $F/N$. Let $X = (X_{ij})$ be a generic matrix of the same size $r \times n$. We denote by $I_s(X)$ the ideal in the polynomial ring $A[X] = A[X_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n]$ generated by the $s$-minors of $X$. Let $B = A[X]_{(m,X)}$ be the ring localized at the graded maximal ideal $(m, X)$ of $A[X]$. The substitution map $A[X] \to A$ where $X_{ij} \mapsto c_{ij}$ now induces a map $\varphi : B \to A$. We consider the ring $A$ as a $B$-algebra via the map $\varphi$. Let

$$b = \text{Ker } \varphi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)B.$$

Set $G = B^r$, and let $L$ denote the submodule $\text{Im}(B^n \xrightarrow{X} G)$ of $G$. Let $G_\nu$ and $L_\nu$ be the homogeneous components of degree $\nu$ of the graded algebras $S_B(G)$ and $\mathcal{R}(L)$, respectively. Then one can check the following.

**Lemma 3.** For any integers $\nu \geq 0$, we have the following:

1. $(G_{\nu+1}/L_{\nu+1}) \otimes_B (B/b) \cong S_{\nu+1}/N_{\nu+1};$
2. $\text{Supp}_B(G_{\nu+1}/L_{\nu+1}) = \text{Supp}_B(B/I_r(X)B);$;
3. The ideal $b$ is generated by a system of parameters of the module $G_{\nu+1}/L_{\nu+1}$.

The following fact concerning $G_{\nu+1}/L_{\nu+1}$ is known by [3, Corollary 3.2] (see also [13, Proposition 3.3]).

**Lemma 4.** For any integer $\nu \geq 0$, we have $G_{\nu+1}/L_{\nu+1}$ is a perfect $B$-module of grade $d$.

The following plays a key role in the proof of Theorem 1. See [11, Proposition 2.4] for the proof.

**Proposition 5.** For any $p \in \text{Min}_B(B/I_r(X)B)$, the equality

$$\ell_B((G_{\nu+1}/L_{\nu+1})_p) = \ell_B((B/I_r(X)B)_p)^{\left(\nu + d + r - 1\right)\over d + r - 1}$$

holds true for all integers $\nu \geq 0$.

3. Proof of Theorem 1

In order to prove Theorem 1, we need to introduce more notation. For any matrix $a$ of size $r \times n$ over an arbitrary ring, we denote by $K_\bullet(a)$ its Eagon-Northcott complex [6]. When $r = 1$, the complex $K_\bullet(a)$ is just the ordinary Koszul complex of the sequence $a$. See [7, Appendix A2] for the definition and more details of complexes of this type. Recall in particular that if $N$ is a parameter module in a free module $F$ as in section 2, then

$$e(F/N) = \chi(K_\bullet(\tilde{N})),$$

where $\chi(K_\bullet(\tilde{N}))$ denotes the Euler-Poincaré characteristic of the complex $K_\bullet(\tilde{N})$ (see [4] and [14]). Moreover, one can check the following by computing $\text{Tor}_B^p(B/IB, A)$ for any $p \geq 0$ (see [5]).
Lemma 6. Using the setting and notation of section 2, we have
\[ \chi(K_\bullet(b) \otimes_B (B/I_r(X)B)) = \chi(K_\bullet(\tilde{N})). \]

Now we can give the proof of Theorem 1.

Proof of Theorem 1. We use the same notation as in section 2. Put \( I = I_r(X) \).

(1): Fix integers \( \nu \geq 0 \). The ideal \( b \) being generated by a system of parameters of the module \( G_{\nu+1}/L^{\nu+1} \), we get
\[
\ell_A(S_{\nu+1}/N^{\nu+1}) \\
= \ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/b)) \\
\geq e(b; G_{\nu+1}/L^{\nu+1}) \\
= \sum_{p \in \Assh_B((G_{\nu+1}/L^{\nu+1}))} e(b; B/p) \cdot \ell_{B_p}((G_{\nu+1}/L^{\nu+1})_p) \\
= \sum_{p \in \Assh_B(B/IB)} e(b; B/p) \cdot \ell_{B_p}((B/IB)_p) \left( \frac{\nu + d + r - 1}{d + r - 1} \right) \\
= e(b; B/IB) \left( \frac{\nu + d + r - 1}{d + r - 1} \right) \\
= \chi(K_\bullet(b) \otimes_B (B/IB)) \left( \frac{\nu + d + r - 1}{d + r - 1} \right) \\
= \chi(K_\bullet(\tilde{N})) \left( \frac{\nu + d + r - 1}{d + r - 1} \right) \\
= e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)
\]
as desired, where \( e(b; *) \) denotes the multiplicity of * with respect to \( b \).

(2): The other implication being clear, by the ideal case, for example, it is enough to show that (ii) implies (i). Assume thus that
\[
\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)
\]
for some \( \nu \geq 0 \). The above argument then gives
\[
\ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/b)) = e(b; G_{\nu+1}/L^{\nu+1}).
\]
It follows that \( G_{\nu+1}/L^{\nu+1} \) is a Cohen-Macaulay \( B \)-module of dimension \( r n \) ([2, (5.12) Corollary]). By Lemma 4, \( G_{\nu+1}/L^{\nu+1} \) is a perfect \( B \)-module of grade \( d \). Thus, by the Auslander-Buchsbaum formula,
\[
\text{depth } B = \text{depth}_B(G_{\nu+1}/L^{\nu+1}) + \text{pd}_B(G_{\nu+1}/L^{\nu+1}) \\
= \dim_B(G_{\nu+1}/L^{\nu+1}) + \text{grade}_B(G_{\nu+1}/L^{\nu+1}) \\
= r n + d \\
= \dim B.
\]
Therefore \( B \) is Cohen-Macaulay so that \( A \) is Cohen-Macaulay, too. \( \square \)
REFERENCES


DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND TECHNOLOGY
MEIJI UNIVERSITY
1-1-1 HIGASHIMITA, TAMA-KU, KAWASAKI 214–8571, JAPAN
E-mail address: hayasaka@isc.meiji.ac.jp